Elementary Row Operations

Elementary Row Operations

Elementary row operations:

Replace: replace one row by the sum of that row and a constant multiple of another row.

Swap: interchange two rows

Scale: multiply a row by a non-zero constant

Let's use the elementary row operations to reduce a matrix to row echelon form.

$$a = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 0 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix};$$

```
a[[2]] = a[[2]] - 2 a[[1]];
a // MatrixForm
```

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -10 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix}$$

```
a[[3]] = a[[3]] + 4 a[[1]];
a // MatrixForm
```

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -10 & 8 \\ 0 & -3 & 13 & -9 \end{pmatrix}$$

```
a[[3]] = a[[3]] + 3 / 4 a[[2]];
a // MatrixForm
```

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -10 & 8 \\ 0 & 0 & \frac{11}{2} & -3 \end{pmatrix}$$

Elementary Matrices

Define procedures which create **elementary matrices**.
Premultiplication of a given matrix by an elementary matrix effects an **elementary row operation**.

```
eReplace[n_, i_, a_, j_] :=
 Module[{e = IdentityMatrix[n]},
  e[[i]] = e[[i]] + a e[[j]]; e]

eSwap[n_, i_, j_] :=
 Module[{e = IdentityMatrix[n], temp},
   temp = e[[i]]; e[[i]] = e[[j]]; e[[j]] = temp; e]

eScale[n_, i_, a_] :=
 Module[{e = IdentityMatrix[n]},
  e[[i]] = a e[[i]]; e]
```

Testing. Create some elementary matrices.

$$a = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 0 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix};$$

```
eReplace[3, 2, -2, 1] // MatrixForm
eSwap[3, 1, 2] // MatrixForm
eScale[3, 1, -1] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Echelon Form

Use elementary row operations to reduce a matrix to row echelon form.
Premultiplication by elementary matrices accomplishes the transformation.

$$a = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 2 & 0 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{pmatrix};$$

```
e1 = eReplace[3, 2, -2, 1];
e2 = eReplace[3, 3, 4, 1];

e2.e1.a;
% // MatrixForm
```

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -10 & 8 \\ 0 & -3 & 13 & -9 \end{pmatrix}$$

```
e3 = eReplace[3, 3, 3 / 4, 2];

e3.e2.e1.a;
% // MatrixForm
```

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 4 & -10 & 8 \\ 0 & 0 & \frac{11}{2} & -3 \end{pmatrix}$$

Reduced Echelon Form

Use row reduction to solve a system of linear equations.

```
a = ( 1  -2   1   0
      0   2  -8   8
     -4   5   9  -9 );

r = RowReduce[a];
% // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Let's check that.

```
<< LinearAlgebra`MatrixManipulation`
```

```
m = TakeColumns[a, 3];
% // MatrixForm

s = TakeColumns[r, -1];
% // MatrixForm

b = TakeColumns[a, -1];
% // MatrixForm

m.s == b
```

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 29 \\ 16 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 8 \\ -9 \end{pmatrix}$$

```
True
```