Lay Chapter 5,
Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Mathematica has several commands for working with eigenvalues and eigenvectors.

```
Clear[a]

a = (1 2 3
     1 2 3
     1 2 3);
```

```
eigenvals = Eigenvalues[a]

{6, 0, 0}
```

Since the Eigenvectors command returns a list eigenvectors, those eigenvectors will appear as rows of a matrix. It is frequently convenient to reassemble them as columns of a matrix.

```
eigenvecsRows = Eigenvectors[a];
% // MatrixForm

eigenvecs = Transpose[eigenvecsRows];
% // MatrixForm
```

$$\begin{pmatrix} 1 & 1 & 1 \\ -3 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The Eigensystem command returns both eigenvalues and eigenvectors.

```
Eigensystem[a];
Map[MatrixForm, %]
```

$$\left\{\begin{pmatrix}6\\0\\0\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\-3 & 0 & 1\\-2 & 1 & 0\end{pmatrix}\right\}$$

Diagonalization of a matrix, $A == P^{-1}.\Lambda.P$

An important result from linear algebra takes the form

```
(* P.A.P^-1 == Λ and A == P^-1.Λ.P *)
```

when the matrix A is diagonalizable. Let's work towards that result.

```
diagλ = DiagonalMatrix[eigenvals];
% // MatrixForm
```

$$\begin{pmatrix}6 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

Now, observe.

```
a.eigenvecs;
% // MatrixForm
```

$$\begin{pmatrix}6 & 0 & 0\\6 & 0 & 0\\6 & 0 & 0\end{pmatrix}$$

```
eigenvecs.diagλ;
% // MatrixForm
```

$$\begin{pmatrix}6 & 0 & 0\\6 & 0 & 0\\6 & 0 & 0\end{pmatrix}$$

Therefore,

```
Inverse[eigenvecs].a.eigenvecs;
% // MatrixForm

diagλ == Inverse[eigenvecs].a.eigenvecs
```

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
True
```

and

```
eigenvecs.diagλ.Inverse[eigenvecs];
% // MatrixForm

a == eigenvecs.diagλ.Inverse[eigenvecs]
```

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

```
True
```

Another Matrix Diagonalization, $A == P^{-1}.\Lambda.P$

```
Clear[a]
```

$$a = \begin{pmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{pmatrix};$$

```
eigenvals = Eigenvalues[a]
```

```
{5, -2, -2, 1}
```

```
eigenvecsRows = Eigenvectors[a];
% // MatrixForm

eigenvecs = Transpose[eigenvecsRows];
% // MatrixForm
```

$$\begin{pmatrix} 2 & 1 & -1 & 2 \\ 6 & -3 & 0 & 4 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & -7 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 6 & 1 & 2 \\ 1 & -3 & 1 & -1 \\ -1 & 0 & 1 & -7 \\ 2 & 4 & 0 & 2 \end{pmatrix}$$

```
Eigensystem[a];
Map[MatrixForm, %]
```

$$\left\{ \begin{pmatrix} 5 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 & 2 \\ 6 & -3 & 0 & 4 \\ 1 & 1 & 1 & 0 \\ 2 & -1 & -7 & 2 \end{pmatrix} \right\}$$

```
diagλ = DiagonalMatrix[eigenvals];
% // MatrixForm
```

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, observe.

```
a.eigenvecs;
% // MatrixForm
```

$$\begin{pmatrix} 10 & -12 & -2 & 2 \\ 5 & 6 & -2 & -1 \\ -5 & 0 & -2 & -7 \\ 10 & -8 & 0 & 2 \end{pmatrix}$$

```
eigenvecs.diagλ;
% // MatrixForm
```

$$\begin{pmatrix} 10 & -12 & -2 & 2 \\ 5 & 6 & -2 & -1 \\ -5 & 0 & -2 & -7 \\ 10 & -8 & 0 & 2 \end{pmatrix}$$

Therefore,

```
Inverse[eigenvecs].a.eigenvecs;
% // MatrixForm

diagλ == Inverse[eigenvecs].a.eigenvecs
```

$$\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
True
```

and

```
eigenvecs.diagλ.Inverse[eigenvecs];
% // MatrixForm

a == eigenvecs.diagλ.Inverse[eigenvecs]
```

$$\begin{pmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{pmatrix}$$

```
True
```

Characteristic Polynomial

Calculate the characteristic polynomial of a certain square matrix.

```
Clear[a, λ, id]

a = (1 2
     3 4);
id = IdentityMatrix[2];

ch[λ_] = Det[a - λ id]
```

$$-2 - 5\,\lambda + \lambda^2$$

Roots of ch[λ] are eigenvalues of a.

```
Solve[ch[λ] == 0, λ] // Flatten
```

$$\left\{\lambda \to \frac{1}{2}\left(5 - \sqrt{33}\right),\ \lambda \to \frac{1}{2}\left(5 + \sqrt{33}\right)\right\}$$

Complex Eigenvalues

```
Clear[a]

a = (1  -2
     1   3);
```

```
eigenvals = Eigenvalues[a]
```

$\{2 + i, 2 - i\}$

```
eigenvecsRows = Eigenvectors[a];
% // MatrixForm

eigenvecs = Transpose[eigenvecsRows];
% // MatrixForm
```

$$\begin{pmatrix} -1 + i & 1 \\ -1 - i & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 + i & -1 - i \\ 1 & 1 \end{pmatrix}$$

```
Eigensystem[a];
Map[MatrixForm, %]
```

$$\left\{\begin{pmatrix} 2 + i \\ 2 - i \end{pmatrix}, \begin{pmatrix} -1 + i & 1 \\ -1 - i & 1 \end{pmatrix}\right\}$$

```
diagλ = DiagonalMatrix[eigenvals];
% // MatrixForm
```

$$\begin{pmatrix} 2 + i & 0 \\ 0 & 2 - i \end{pmatrix}$$

Now, observe.

```
a.eigenvecs;
% // MatrixForm
```

$$\begin{pmatrix} -3+i & -3-i \\ 2+i & 2-i \end{pmatrix}$$

```
eigenvecs.diagλ;
% // MatrixForm
```

$$\begin{pmatrix} -3+i & -3-i \\ 2+i & 2-i \end{pmatrix}$$

Therefore,

```
Inverse[eigenvecs].a.eigenvecs;
% // MatrixForm

diagλ == Inverse[eigenvecs].a.eigenvecs
```

$$\begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$$

```
True
```

and

```
eigenvecs.diagλ.Inverse[eigenvecs];
% // MatrixForm

a == eigenvecs.diagλ.Inverse[eigenvecs]
```

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

```
True
```

Polar Form of a Matrix

```
Clear[m, a, b, φ, rot, r]

a = √3;
b = 1;

m = (a  -b
     b   a);
% // MatrixForm
```

$$\begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$

Find the angle of the rotation.

```
φ = ArcTan[b / a]
```

$$\frac{\pi}{6}$$

... and compute the rotation matrix.

```
rot[φ_] := (Cos[φ]  -Sin[φ]
            Sin[φ]   Cos[φ]);

rot[φ];
% // MatrixForm
```

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Find the scaling factor.

```
r = √(a² + b²)
```

2

Check the decomposition.

```
m == r rot[φ]

True
```

The Rotation Due to a Complex Eigenvalue

If a real 2x2 matrix has a complex eigenvalue, a particular decomposition will display the rotation "hidden within." See Lay 5.5, pp. 338--341.

```
Clear[m, a, b, c, p];

m = (1  -2
     1   3);
% // MatrixForm

(1  -2
 1   3)
```

```
evals = Eigenvalues[m]

{2 + i, 2 - i}
```

Construct the matrix c.

```
a = Re[evals[[1]]];
b = -Im[evals[[1]]];

c = (a  -b
     b   a);
% // MatrixForm

( 2   1
 -1   2)
```

Construct the matrix p.

```
evecs = Eigenvectors[m]

{{-1 + i, 1}, {-1 - i, 1}}
```

```
p1 = Re[evecs[[1]]];
p2 = Im[evecs[[1]]];

p = Transpose[{p1, p2}];
% // MatrixForm
```

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Check these results.
We should find that m = p.c.Inverse[p] where c is a polar matrix (rotation with scaling).

```
p.c.Inverse[p];
% // MatrixForm

m == p.c.Inverse[p]
```

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

```
True
```