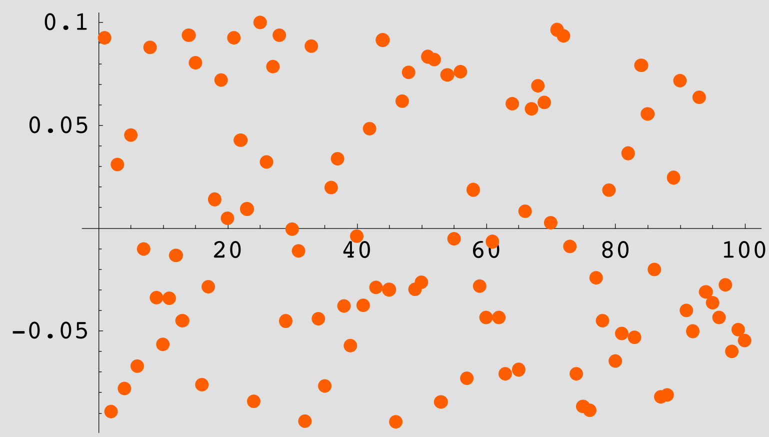
Fitting Data

Noisy Data

Mathematica's Random function can be used to create "noisy" data.
Each time the following cell is evaluated, a different picture is produced.

```
ϵ = 0.1;
nums = Table[Random[Real, {-ϵ, ϵ}],
   {k, 100}];

ListPlot[nums,
  PlotStyle → {OrangeRed, PointSize[0.02]}];
```



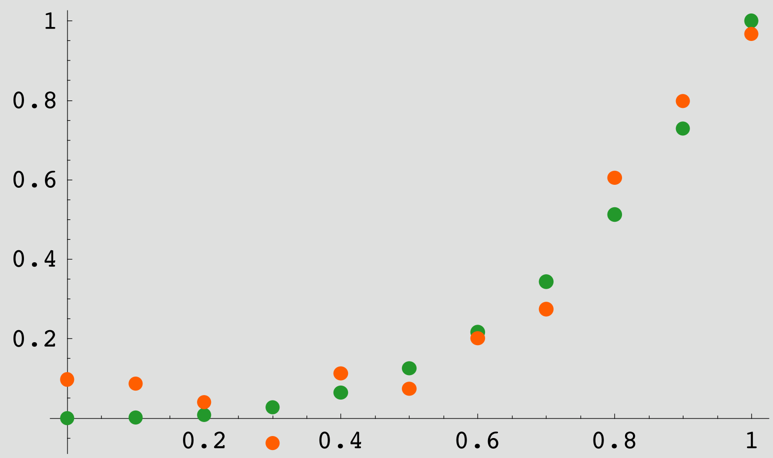
Let's create some noisy data, and then pretend that we have just discovered it somewhere.

```
Clear[f, x, xs, ys, noisyYs, data, noisyData]

f[x_] = x^3;

xs = 0.1 Range[0, 10];
ys = Map[f, xs];

noisyYs = Map[# + Random[Real, {-ϵ, ϵ}] &, ys];

data = Transpose[{xs, ys}];
noisyData = Transpose[{xs, noisyYs}];
```

Oh, my. Look at this. It's some noisy data.

```
<< Graphics`Graphics`
```

```
DisplayTogether[
  ListPlot[data,
   PlotStyle → {ForestGreen, PointSize[0.02]}],
  ListPlot[noisyData,
   PlotStyle → {OrangeRed, PointSize[0.02]}]];
```



Fitting Data with a Regression Line
Lay 6.5, Example 1, page 420

Find a least-squares line that best fits a set of data points.

```
Clear[data, X, x1, x2, y, β, β0, β1, x];

data = {{2, 1}, {5, 2}, {7, 3}, {8, 3}};
% // MatrixForm
```

$$\begin{pmatrix} 2 & 1 \\ 5 & 2 \\ 7 & 3 \\ 8 & 3 \end{pmatrix}$$

Construct the equation X.β=y.
A typical row of X.β is of the form 1*β0 + x*β1

```
x1 = {1, 1, 1, 1};
x2 = Transpose[data][[1]];
X = Transpose[{x1, x2}];
β = {β0, β1};
y = Transpose[data][[2]];

Print[X // MatrixForm, ".", β // MatrixForm, "=", y // MatrixForm]
```

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 4 & 22 \\ 22 & 142 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 9 \\ 57 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " = ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 4 & 22 \\ 22 & 142 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 57 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {β0, β1}][[1]]
```

$$\left\{\beta 0 \to \frac{2}{7}, \beta 1 \to \frac{5}{14}\right\}$$

```
β = β /. soln
```

$\left\{\frac{2}{7}, \frac{5}{14}\right\}$

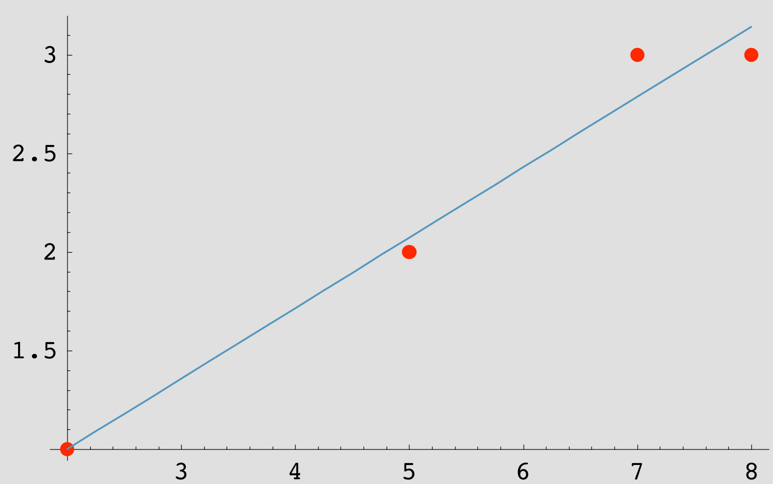
Interpret the result: The (least squares) line which best fits the data is

```
Clear[y, x];

y[x_] = β.{1, x}
```

$\frac{2}{7} + \frac{5x}{14}$

Now plot the data and its regression line.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

linePlot = Plot[y[x], {x, 2, 8},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, linePlot, DisplayFunction → $DisplayFunction];
Print["regression line: y = ", β[[1]], " + ", β[[2]], "x."]
```



regression line: y = $\frac{2}{7}$ + $\frac{5}{14}$x.

Fitting Data with a Regression Line
Lay 6.6.1

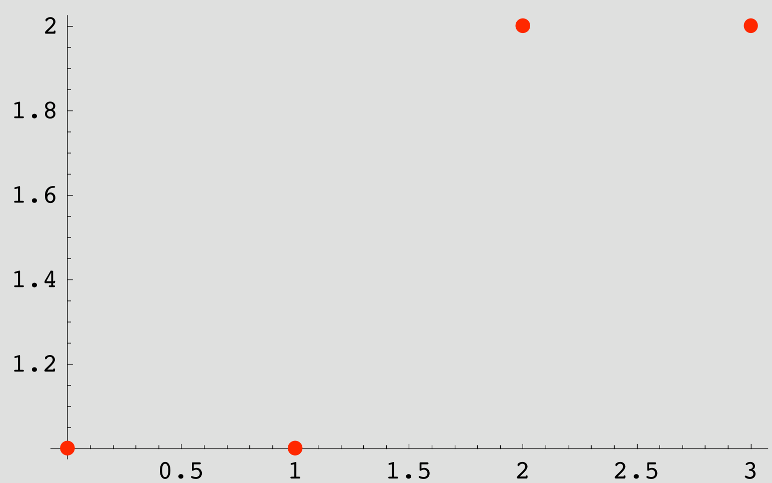
Find a least-squares line that best fits a set of data points.

```
Clear[data, X, x1, x2, y, β, β0, β1, βHatx];

data = {{0, 1}, {1, 1}, {2, 2}, {3, 2}};
% // MatrixForm
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \\ 3 & 2 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the equation X.β=y.
A typical row of X.β is of the form 1*β0 + x*β1

```
x1 = {1, 1, 1, 1};
x2 = Transpose[data][[1]];
X = Transpose[{x1, x2}];
β = {β0, β1};
y = Transpose[data][[2]];

Print[X // MatrixForm, ".", β // MatrixForm, "=", y // MatrixForm]
```

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

Now solve the system $X^T \mathrm{X}\beta = X^T \mathrm{y}$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " = ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {β0, β1}][[1]]
```

$$\left\{\beta 0 \to \frac{9}{10}, \beta 1 \to \frac{2}{5}\right\}$$

```
βHat = β /. soln
```

$\left\{\frac{9}{10}, \frac{2}{5}\right\}$

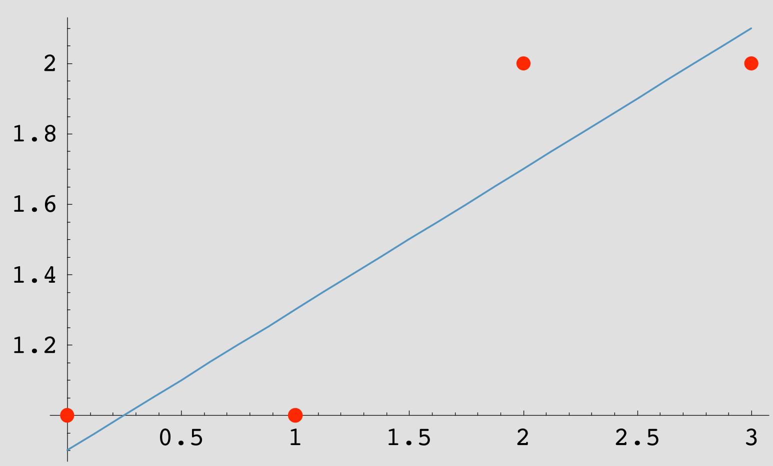
Interpret the result: The (least squares) line which best fits the data is

```
Clear[y, x];

y[x_] = βHat.{1, x}
```

$\frac{9}{10} + \frac{2\,x}{5}$

Now plot the data and its regression line.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

linePlot = Plot[y[x], {x, 0, 3},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, linePlot, DisplayFunction → $DisplayFunction];
Print["regression line: y = ", βHat[[1]], " + ", βHat[[2]], "x."]
```



regression line: y = $\frac{9}{10}$ + $\frac{2}{5}$x.

Fitting Data with a Regression Line
Lay 6.6.3

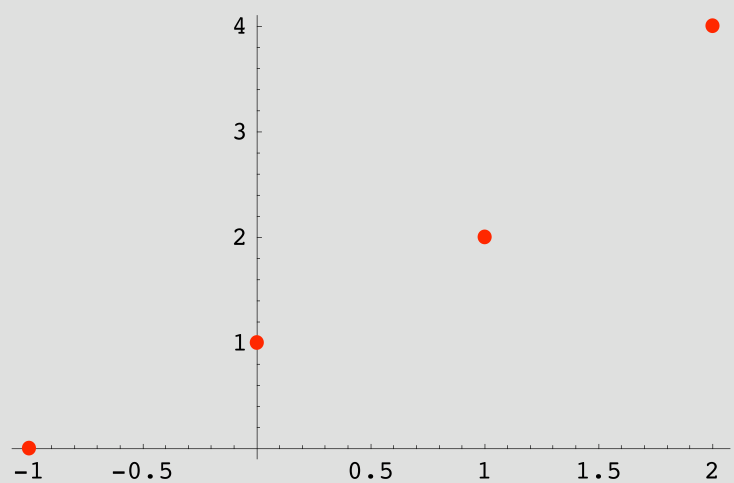
Find a least-squares line that best fits a set of data points.

```
Clear[data, X, x1, x2, y, β, β0, β1, βHatx];

data = {{-1, 0}, {0, 1}, {1, 2}, {2, 4}};
% // MatrixForm
```

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the equation X.β=y.
A typical row of X.β is of the form 1*β0 + x*β1

```
x1 = {1, 1, 1, 1};
x2 = Transpose[data][[1]];
X = Transpose[{x1, x2}];
β = {β0, β1};
y = Transpose[data][[2]];

Print[X // MatrixForm, ".", β // MatrixForm, "=", y // MatrixForm]
```

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " = ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {β0, β1}][[1]]
```

$$\left\{\beta 0 \to \frac{11}{10}, \beta 1 \to \frac{13}{10}\right\}$$

```
βHat = β /. soln
```

$\left\{\frac{11}{10}, \frac{13}{10}\right\}$

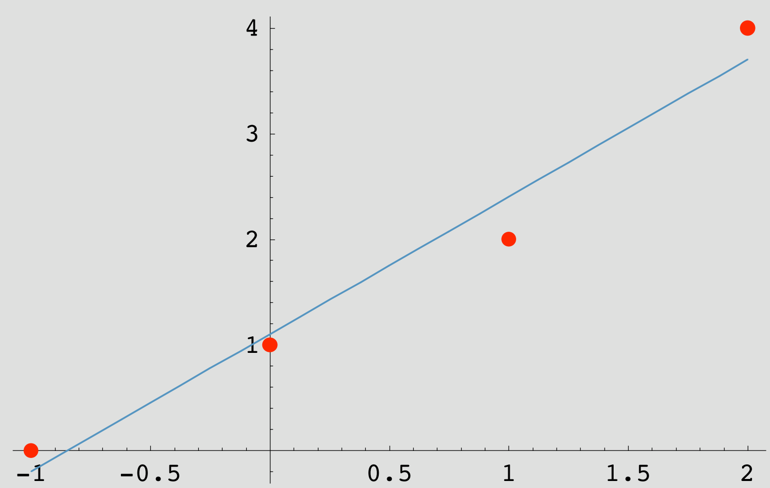
Interpret the result: The (least squares) line which best fits the data is

```
Clear[y, x];

y[x_] = βHat.{1, x}
```

$\frac{11}{10} + \frac{13\,x}{10}$

Now plot the data and its regression line.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

linePlot = Plot[y[x], {x, -1, 2},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, linePlot, DisplayFunction → $DisplayFunction];
Print["regression line: y = ", βHat[[1]], " + ", βHat[[2]], "x."]
```



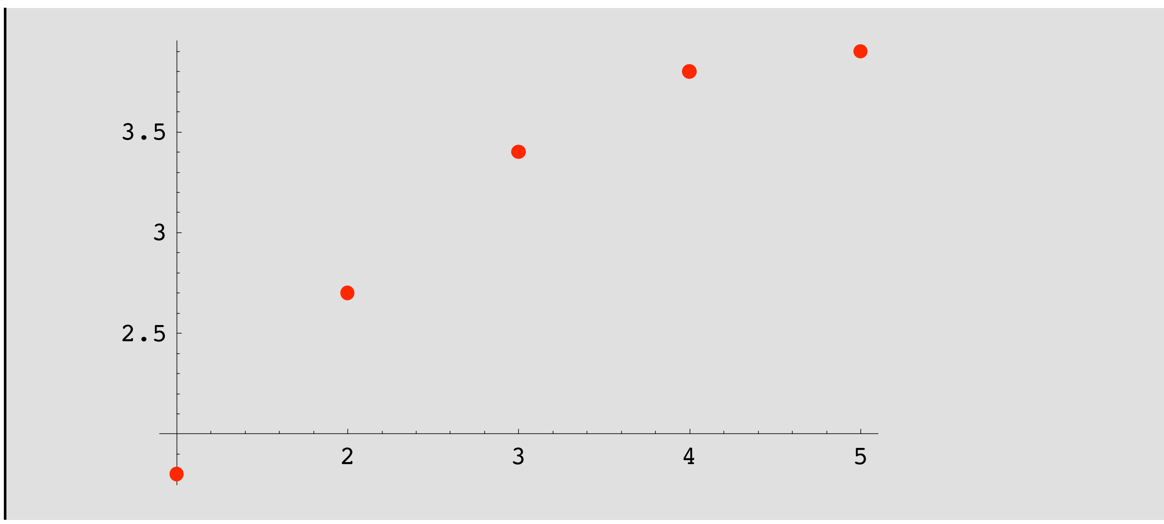
regression line: y = $\frac{11}{10} + \frac{13}{10}$x.

Fitting Data with a Quadratic Polynomial
Lay 6.6.7

Find a quadratic polynomial of the form y = $\beta 1$ x + $\beta 2$ x^2 that best fits the data.

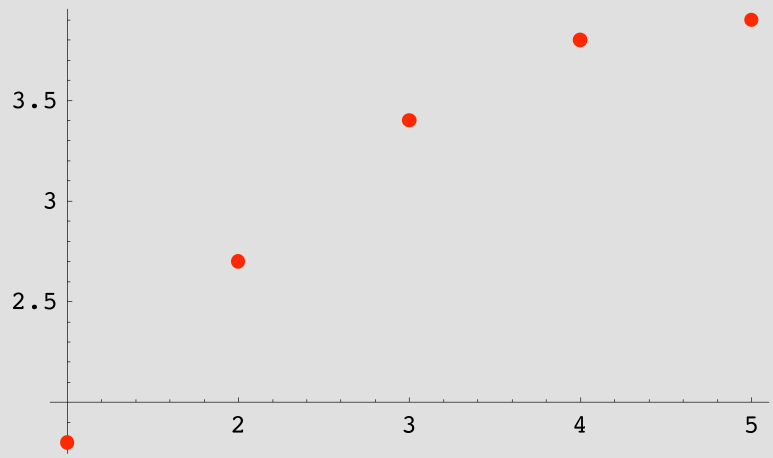
```
Clear[data, X, x1, x2, y, β, β1, β2, βHatx];

data = {{1, 1.8}, {2, 2.7}, {3, 3.4}, {4, 3.8}, {5, 3.9}};
% // MatrixForm

dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```

$$\begin{pmatrix} 1 & 1.8 \\ 2 & 2.7 \\ 3 & 3.4 \\ 4 & 3.8 \\ 5 & 3.9 \end{pmatrix}$$



Visualize the data.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ**.
A typical row of X.β is of the form $\beta 1 x + \beta 2\, x^2$

```
x1 = Transpose[data][[1]];
x2 = x1^2;
X = Transpose[{x1, x2}];
β = {β1, β2};
y = Transpose[data][[2]];
ϵ = {ϵ1, ϵ2, ϵ3, ϵ4, ϵ5};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", β // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{pmatrix} . \begin{pmatrix} \beta 1 \\ \beta 2 \end{pmatrix} + \begin{pmatrix} \epsilon 1 \\ \epsilon 2 \\ \epsilon 3 \\ \epsilon 4 \\ \epsilon 5 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " == ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 55 & 225 \\ 225 & 979 \end{pmatrix} . \begin{pmatrix} \beta 1 \\ \beta 2 \end{pmatrix} == \begin{pmatrix} 52.1 \\ 201.5 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {β1, β2}][[1]]
```

```
{β1 → 1.76037, β2 → -0.198758}
```

```
βHat = β /. soln
```

```
{1.76037, -0.198758}
```

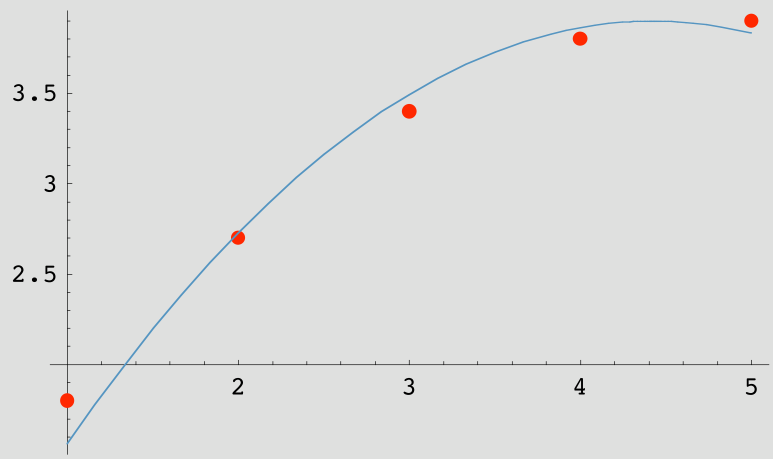
Interpret the result: The (least squares) quadratic polynomial we seek is

```
Clear[y, x];

y[x_] = βHat.{x, x^2}
```

$1.76037\,x - 0.198758\,x^2$

Now plot the data and its least squares polynomial.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

curvePlot = Plot[y[x], {x, 1, 5},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["regression polynomial: y = ", βHat[[1]], "x ", βHat[[2]], "x²."]
```




```
regression polynomial: y = 1.76037x -0.198758x².
```

Fitting Data with a Trigonometric Sum
Lay 6.6.9

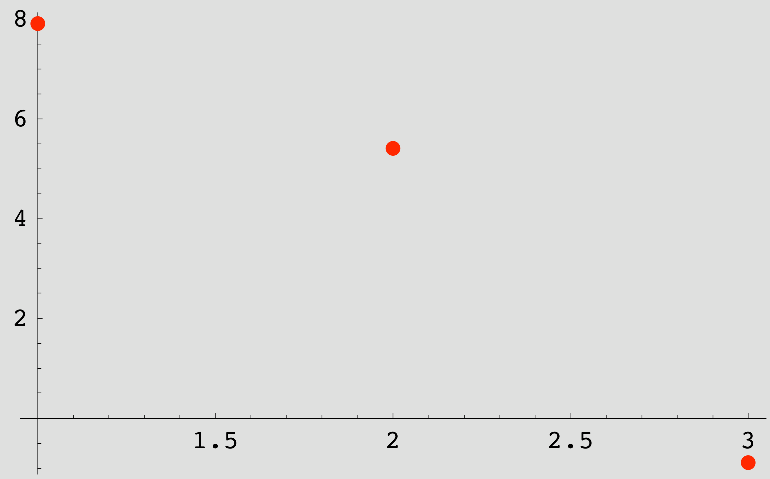
Find a function of the form y =A Cos[x] + B Sin[x] that best fits the data.

```
Clear[data, X, x1, x2, y, β, A, B, βHat, ϵ];

data = {{1, 7.9}, {2, 5.4}, {3, -.9}};
% // MatrixForm
```

$$\begin{pmatrix} 1 & 7.9 \\ 2 & 5.4 \\ 3 & -0.9 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ.**
A typical row of X.β is of the form A Cos[x] + B Sin[x]

```
x1 = Map[Cos, Transpose[data][[1]]];
x2 = Map[Sin, Transpose[data][[1]]];
X = Transpose[{x1, x2}];
β = {A, B};
y = Transpose[data][[2]];
ϵ = {ϵ1, ϵ2, ϵ3};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", β // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 7.9 \\ 5.4 \\ -0.9 \end{pmatrix} = \begin{pmatrix} \text{Cos}[1] & \text{Sin}[1] \\ \text{Cos}[2] & \text{Sin}[2] \\ \text{Cos}[3] & \text{Sin}[3] \end{pmatrix} . \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \epsilon 1 \\ \epsilon 2 \\ \epsilon 3 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[1]^2 + \text{Cos}[2]^2 + \text{Cos}[3]^2 & \text{Cos}[1]\,\text{Sin}[1] + \text{Cos}[2]\,\text{Sin}[2] + \text{Co} \\ \text{Cos}[1]\,\text{Sin}[1] + \text{Cos}[2]\,\text{Sin}[2] + \text{Cos}[3]\,\text{Sin}[3] & \text{Sin}[1]^2 + \text{Sin}[2]^2 + \text{Sin}[3]^2 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

```
( 2.91219 )
( 11.4308 )
```

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " == ", Transpose[X].y // MatrixForm]
```

```
( Cos[1]^2 + Cos[2]^2 + Cos[3]^2                 Cos[1] Sin[1] + Cos[2] Sin[2] + Cc
  Cos[1] Sin[1] + Cos[2] Sin[2] + Cos[3] Sin[3]   Sin[1]^2 + Sin[2]^2 + Sin[3]^2
```

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {A, B}][[1]]
```

```
{A → 2.34212, B → 7.4475}
```

```
βHat = β /. soln
```

```
{2.34212, 7.4475}
```

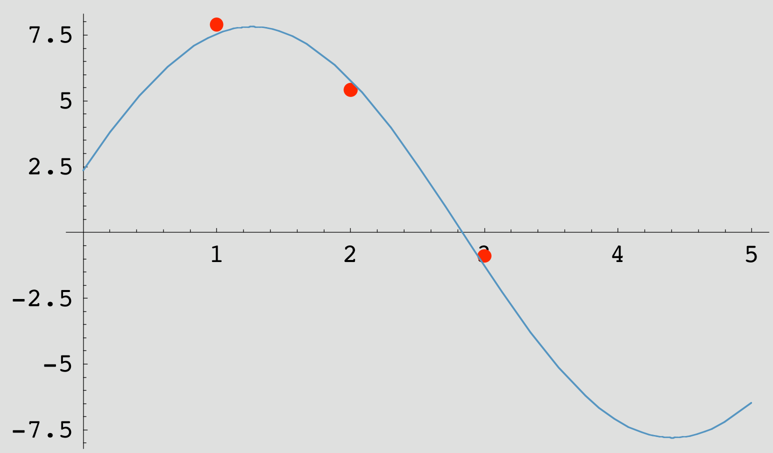
Interpret the result: The (least squares) quadratic polynomial we seek is

```
Clear[y, x];

y[x_] = βHat.{Cos[x], Sin[x]}
```

```
2.34212 Cos[x] + 7.4475 Sin[x]
```

Now plot the data and its least squares polynomial.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

curvePlot = Plot[y[x], {x, 0, 5},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["regression polynomial: y[x] = ", y[x]]
```




```
regression polynomial: y[x] = 2.34212 Cos[x] + 7.4475 Sin[x]
```

Fitting Data with a Sum of Exponentials
Lay 6.6.10

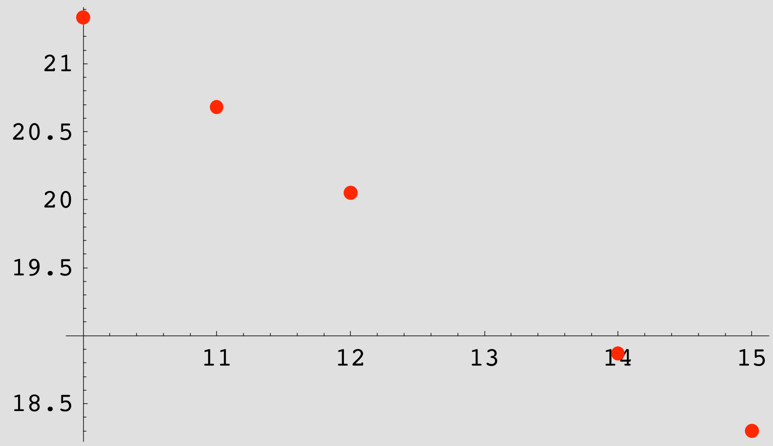
Find a function of the form y = $M_A\, e^{-.02\, t} + M_B\, e^{-.07\, t}$ which best fits the data.

```
Clear[data, X, x1, x2, y, β, A, B, βHat, ϵ, f, g, t];

data = {{10, 21.34}, {11, 20.68}, {12, 20.05}, {14, 18.87}, {15, 18.30}};
% // MatrixForm
```

$$\begin{pmatrix} 10 & 21.34 \\ 11 & 20.68 \\ 12 & 20.05 \\ 14 & 18.87 \\ 15 & 18.3 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ**.
A typical row of X.β is of the form A Cos[x] + B Sin[x]

```
f[t_] = Exp[-.02 t];
g[t_] = Exp[-.07 t];

x1 = Map[f, Transpose[data][[1]]];
x2 = Map[g, Transpose[data][[1]]];
X = Transpose[{x1, x2}];
β = {Ma, Mb};
y = Transpose[data][[2]];
ϵ = {ϵ1, ϵ2, ϵ3, e4, e5};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", β // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 21.34 \\ 20.68 \\ 20.05 \\ 18.87 \\ 18.3 \end{pmatrix} = \begin{pmatrix} 0.818731 & 0.496585 \\ 0.802519 & 0.463013 \\ 0.786628 & 0.431711 \\ 0.755784 & 0.375311 \\ 0.740818 & 0.349938 \end{pmatrix} . \begin{pmatrix} Ma \\ Mb \end{pmatrix} + \begin{pmatrix} \epsilon1 \\ \epsilon2 \\ \epsilon3 \\ e4 \\ e5 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 3.05316 & 1.66064 \\ 1.66064 & 0.910667 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 77.6583 \\ 42.314 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 β // MatrixForm, " == ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 3.05316 & 1.66064 \\ 1.66064 & 0.910667 \end{pmatrix} . \begin{pmatrix} Ma \\ Mb \end{pmatrix} == \begin{pmatrix} 77.6583 \\ 42.314 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.β == Transpose[X].y, {Ma, Mb}][[1]]
```

{Ma → 19.9411, Mb → 10.1015}

```
βHat = β /. soln
```

{19.9411, 10.1015}

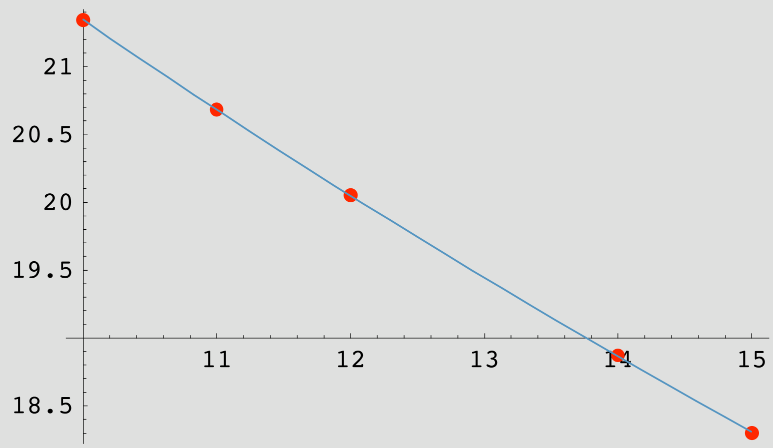
Interpret the result: The (least squares) function we seek is

```
Clear[y, x];

y[t_] = βHat.{f[t], g[t]}
```

$10.1015\, e^{-0.07\, t} + 19.9411\, e^{-0.02\, t}$

Now plot the data and its least squares polynomial.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   DisplayFunction → Identity];

curvePlot = Plot[y[t], {t, 10, 15},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["regression function: y[x] = ", y[x]]
```




```
regression function: y[x] = 10.1015 ⅇ^(-0.07 x) + 19.9411 ⅇ^(-0.02 x)
```

Orbit of a Comet
Lay 6.6.11

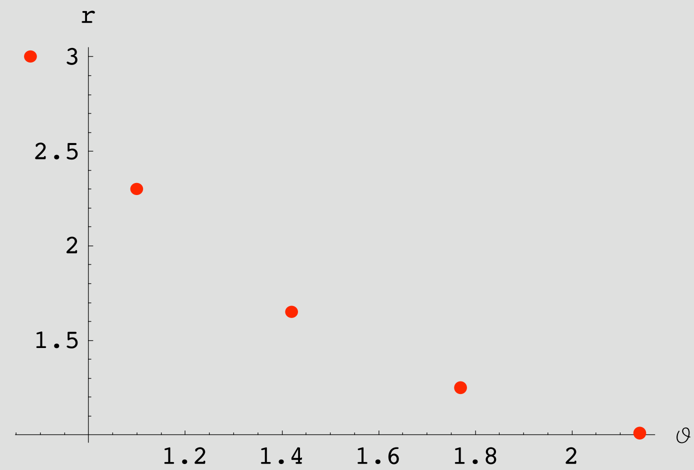
Find a function of the form $y = \beta + e\,(r \operatorname{Cos}[\vartheta])$ that best fits the astronomical data.

```
Clear[data, y, X, β, βHat, x, ϵ, r, e];

data =
  {{.88, 3.00}, {1.10, 2.30}, {1.42, 1.65}, {1.77, 1.25}, {2.14, 1.01}};
% // MatrixForm
```

$$\begin{pmatrix} 0.88 & 3. \\ 1.1 & 2.3 \\ 1.42 & 1.65 \\ 1.77 & 1.25 \\ 2.14 & 1.01 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   AxesLabel → {ϑ, r},
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ**.
A typical row of X.β is of the form β*1 + e* (r Cos[ϑ])

```
x1 = {1, 1, 1, 1, 1};
rs = Transpose[data][[2]];
ϑs = Transpose[data][[1]];
cosϑs = Cos[ϑs];
x2 = rs cosϑs;
X = Transpose[{x1, x2}];

βvec = {β, e};
y = rs;
ϵ = {ϵ1, ϵ2, ϵ3, ϵ4, ϵ5};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", βvec // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 3. \\ 2.3 \\ 1.65 \\ 1.25 \\ 1.01 \end{pmatrix} = \begin{pmatrix} 1 & 1.91145 \\ 1 & 1.04327 \\ 1 & 0.247872 \\ 1 & -0.247361 \\ 1 & -0.544351 \end{pmatrix} . \begin{pmatrix} \beta \\ e \end{pmatrix} + \begin{pmatrix} \epsilon1 \\ \epsilon2 \\ \epsilon3 \\ \epsilon4 \\ \epsilon5 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 5. & 2.41088 \\ 2.41088 & 5.16101 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 9.21 \\ 7.68388 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // MatrixForm, ".",
 βvec // MatrixForm, " = ", Transpose[X].y // MatrixForm]
```

$$\begin{pmatrix} 5. & 2.41088 \\ 2.41088 & 5.16101 \end{pmatrix} . \begin{pmatrix} \beta \\ e \end{pmatrix} = \begin{pmatrix} 9.21 \\ 7.68388 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.βvec == Transpose[X].y, {β, e}][[1]]
```

{β → 1.45093, e → 0.811053}

```
βHat = {β, e} = {β, e} /. soln
```

{1.45093, 0.811053}

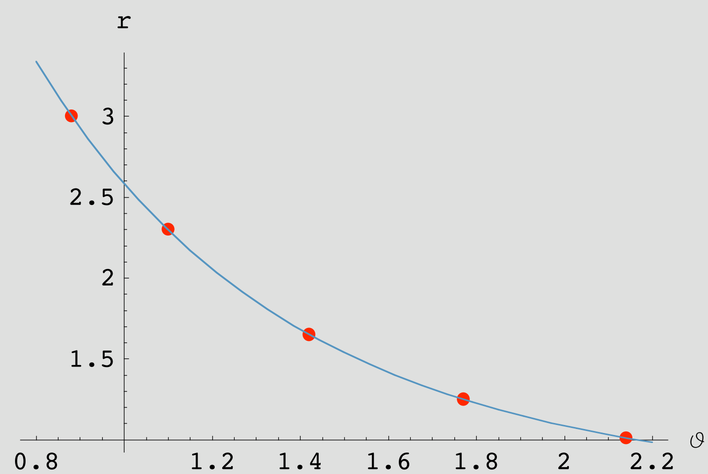
Interpret the result:
Since 0 < e < 1, **the orbit is elliptic**.
The (least squares) function we seek is

```
Clear[r, ϑ];

r[ϑ_] = β / (1 - e Cos[ϑ])
```

$$\frac{1.45093}{1 - 0.811053 \text{Cos}[\vartheta]}$$

Now plot the data and its least squares approximation.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

curvePlot = Plot[r[ϑ], {ϑ, .8, 2.2},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["least squares approximation: r[ϑ] = ", r[ϑ]]
```



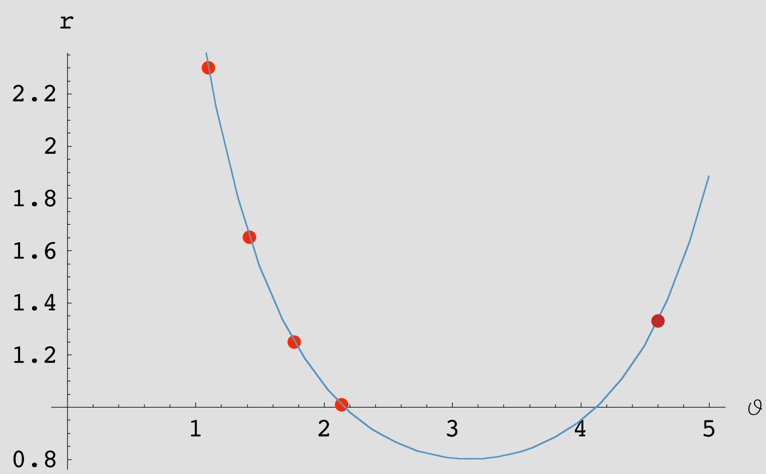
least squares approximation: r[ϑ] = $\frac{1.45093}{1 - 0.811053\,\mathrm{Cos}[\vartheta]}$

```
Print[
 "When ϑ = 4.6, the comet will have radial coordinates (r,ϑ) = (",
 r[4.6] , ",", 4.6, ")."]
```

When ϑ = 4.6, the comet will have radial coordinates (r,ϑ) = (
 1.32995,4.6).

Let's plot that.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

curvePlot = Plot[r[ϑ], {ϑ, .8, 5},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

extraDot = ListPlot[{{4.6, r[4.6]}},
   PlotStyle → {IndianRed, PointSize[.02]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, extraDot, DisplayFunction → $DisplayFunction];
```



Super!

Blood Pressure vs. Weight
Lay 6.6.12

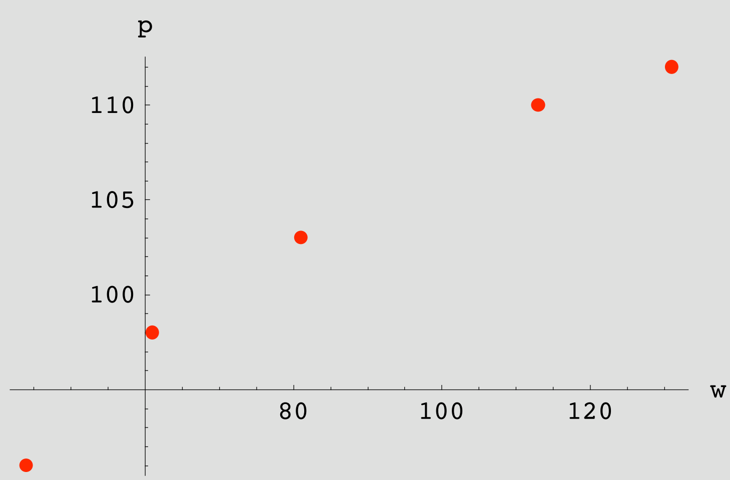
Find a function of the form $p = \beta 0 + \beta 1 \ln(w)$ that best fits the data.

```
Clear[data, y, X, β, β0, β1, βHat, x, ϵ, w, p];

data = {{44, 91}, {61, 98}, {81, 103}, {113, 110}, {131, 112}};
% // MatrixForm
```

$$\begin{pmatrix} 44 & 91 \\ 61 & 98 \\ 81 & 103 \\ 113 & 110 \\ 131 & 112 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   AxesLabel → {w, p},
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ**.
A typical row of X.β is of the form β0*1 + β1*Log[w]

```
x1 = {1, 1, 1, 1, 1};
ps = Transpose[data][[2]];
ws = Transpose[data][[1]];
x2 = Map[Log, ws];
X = Transpose[{x1, x2}];

βvec = {β0, β1};
y = ps;
ϵ = {ϵ1, ϵ2, ϵ3, ϵ4, ϵ5};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", βvec // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 91 \\ 98 \\ 103 \\ 110 \\ 112 \end{pmatrix} = \begin{pmatrix} 1 & \text{Log}[44] \\ 1 & \text{Log}[61] \\ 1 & \text{Log}[81] \\ 1 & \text{Log}[113] \\ 1 & \text{Log}[131] \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} + \begin{pmatrix} \epsilon 1 \\ \epsilon 2 \\ \epsilon 3 \\ \epsilon 4 \\ \epsilon 5 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 5 & \text{Log}[44] + \text{Log}[61] + \text{Log}[81] + \text{I} \\ \text{Log}[44] + \text{Log}[61] + \text{Log}[81] + \text{Log}[113] + \text{Log}[131] & \text{Log}[44]^2 + \text{Log}[61]^2 + \text{Log}[81]^2 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 514 \\ 91\,\text{Log}[44] + 98\,\text{Log}[61] + 103\,\text{Log}[81] + 110\,\text{Log}[113] + 112\,\text{Log}[131] \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // N // MatrixForm, ".",
 βvec // MatrixForm, " = ", Transpose[X].y // N // MatrixForm]
```

$$\begin{pmatrix} 5. & 21.8921 \\ 21.8921 & 96.6463 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \end{pmatrix} = \begin{pmatrix} 514. \\ 2265.89 \end{pmatrix}$$

```
soln = Solve[Transpose[X].X.βvec == Transpose[X].y, {β0, β1}][[1]] // N

{β0 → 17.9243, β1 → 19.385}
```

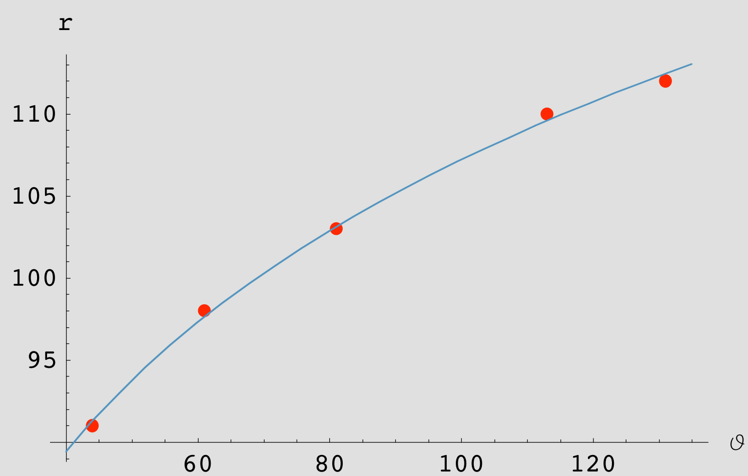
```
βHat = {β0, β1} = {β0, β1} /. soln

{17.9243, 19.385}
```

Interpret the result: The (least squares) function we seek is

```
Clear[y, x];

f[w_] = 1;
g[w_] = Log[w];

y[w_] = βHat.{f[w], g[w]}

17.9243 + 19.385 Log[w]
```

Now plot the data and its least squares approximation.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

curvePlot = Plot[y[w], {w, 40, 135},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["least squares approximation: y[w] = ", y[w]]
```



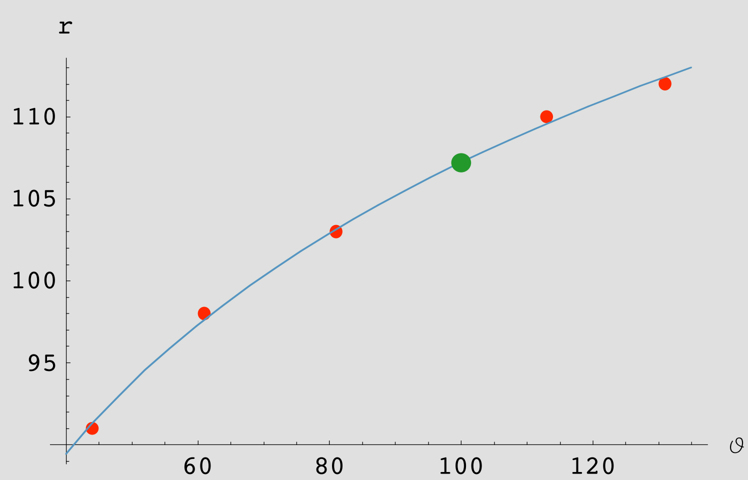

```
least squares approximation: y[w] = 17.9243 + 19.385 Log[w]
```

```
Print["The systolic blood pressure of a healthy
   child weighing 100 lbs should be about ", y[100] , " mm."]
```

```
The systolic blood pressure of a healthy
   child weighing 100 lbs should be about 107.196 mm.
```

Let's plot that.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

curvePlot = Plot[y[w], {w, 40, 135},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

extraDot = ListPlot[{{100, y[100]}},
   PlotStyle → {ForestGreen, PointSize[.03]},
   AxesLabel → {ϑ, r},
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, extraDot, DisplayFunction → $DisplayFunction];
```



Super!

Takeoff Performance of an Airplane
Lay 6.6.13

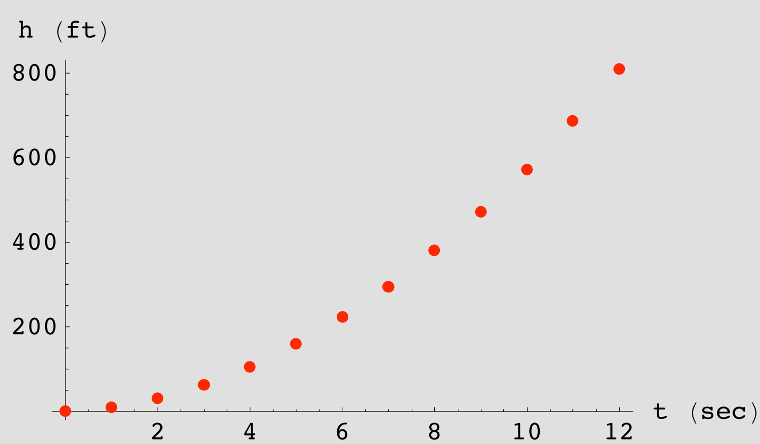
Find a cubic polynomial that best fits the data.

```
Clear[data, y, X, β, β0, β1, β2, β3, βHat, x, ϵ, w, p];

ts = Range[0, 12];
hs = {0, 8.8, 29.9, 62.0, 104.7, 159.1,
   222.0, 294.5, 380.4, 471.1, 571.7, 686.8, 809.2};
data = Transpose[{ts, hs}];
% // MatrixForm
```

$$\begin{pmatrix} 0 & 0 \\ 1 & 8.8 \\ 2 & 29.9 \\ 3 & 62. \\ 4 & 104.7 \\ 5 & 159.1 \\ 6 & 222. \\ 7 & 294.5 \\ 8 & 380.4 \\ 9 & 471.1 \\ 10 & 571.7 \\ 11 & 686.8 \\ 12 & 809.2 \end{pmatrix}$$

Visualize the data.

```
dotPlot = ListPlot[data,
   AxesLabel → {"t (sec)", "h (ft)"},
   PlotStyle → {Red, PointSize[.02]}];
```



Construct the **linear model y = X.β + ϵ**.
A typical row of X.β is of the form $\beta0*1 + \beta1*t + \beta2\, t^2 + \beta3\, t^3$

```
x1 = {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1};
x2 = ts;
x3 = x2^2;
x4 = x2 x3;
X = Transpose[{x1, x2, x3, x4}];

βvec = {β0, β1, β2, β3};
y = hs;
ϵ = {ϵ0, ϵ1, ϵ2, ϵ3, ϵ4, ϵ5, ϵ6, ϵ7, ϵ8, ϵ9, ϵ10, ϵ11, ϵ12};

Print[y // MatrixForm, " = ", X // MatrixForm,
 ".", βvec // MatrixForm, " + ", ϵ // MatrixForm]
```

$$\begin{pmatrix} 0 \\ 8.8 \\ 29.9 \\ 62. \\ 104.7 \\ 159.1 \\ 222. \\ 294.5 \\ 380.4 \\ 471.1 \\ 571.7 \\ 686.8 \\ 809.2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \\ 1 & 7 & 49 & 343 \\ 1 & 8 & 64 & 512 \\ 1 & 9 & 81 & 729 \\ 1 & 10 & 100 & 1000 \\ 1 & 11 & 121 & 1331 \\ 1 & 12 & 144 & 1728 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \\ \beta 2 \\ \beta 3 \end{pmatrix} + \begin{pmatrix} \epsilon 0 \\ \epsilon 1 \\ \epsilon 2 \\ \epsilon 3 \\ \epsilon 4 \\ \epsilon 5 \\ \epsilon 6 \\ \epsilon 7 \\ \epsilon 8 \\ \epsilon 9 \\ \epsilon 10 \\ \epsilon 11 \\ \epsilon 12 \end{pmatrix}$$

Use the normal equations.

```
Transpose[X].X;
% // MatrixForm
```

$$\begin{pmatrix} 13 & 78 & 650 & 6084 \\ 78 & 650 & 6084 & 60710 \\ 650 & 6084 & 60710 & 630708 \\ 6084 & 60710 & 630708 & 6735950 \end{pmatrix}$$

```
Transpose[X].y;
% // MatrixForm
```

$$\begin{pmatrix} 3800.2 \\ 35127.7 \\ 348064. \\ 3.5998\times 10^6 \end{pmatrix}$$

Now solve the system $X^T X\beta = X^T y$.

```
Print[Transpose[X].X // N // MatrixForm, ".",
 βvec // MatrixForm, " = ", Transpose[X].y // N // MatrixForm]
```

$$\begin{pmatrix} 13. & 78. & 650. & 6084. \\ 78. & 650. & 6084. & 60710. \\ 650. & 6084. & 60710. & 630708. \\ 6084. & 60710. & 630708. & 6.73595\times 10^6 \end{pmatrix} . \begin{pmatrix} \beta 0 \\ \beta 1 \\ \beta 2 \\ \beta 3 \end{pmatrix} = \begin{pmatrix} 3800.2 \\ 35127.7 \\ 348064. \\ 3.5998\times 10^6 \end{pmatrix}$$

```
soln =
 Solve[Transpose[X].X.βvec == Transpose[X].y, {β0, β1, β2, β3}][[1]] // N

{β0 → -0.855769, β1 → 4.70249, β2 → 5.55537, β3 → -0.0273601}
```

```
βHat = {β0, β1, β2, β3} = {β0, β1, β2, β3} /. soln

{-0.855769, 4.70249, 5.55537, -0.0273601}
```

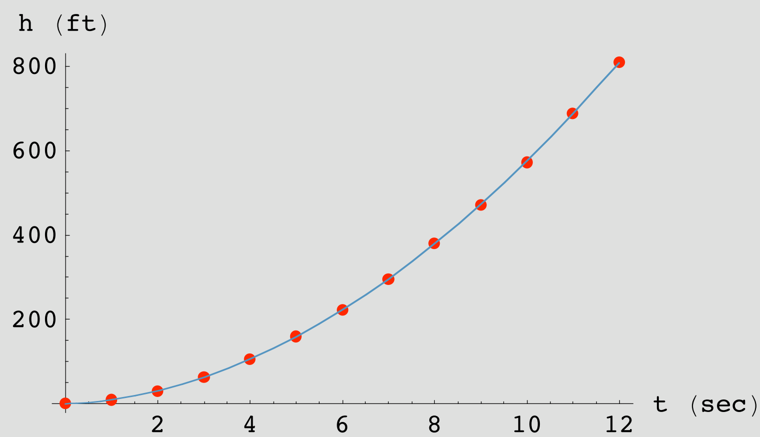
Interpret the result: The (least squares) function we seek is

```
Clear[y, t];

y[t_] = βHat.{1, t, t^2, t^3}
```

$$-0.855769 + 4.70249\, t + 5.55537\, t^2 - 0.0273601\, t^3$$

Now plot the data and its least squares approximation.

```
dotPlot = ListPlot[data,
   PlotStyle → {Red, PointSize[.02]},
   AxesLabel → {"t (sec)", "h (ft)"},
   DisplayFunction → Identity];

curvePlot = Plot[y[t], {t, 0, 12},
   PlotStyle -> SteelBlue,
   DisplayFunction → Identity];

Show[dotPlot, curvePlot, DisplayFunction → $DisplayFunction];
Print["least squares approximation: y[t] = ", y[t]]
```



least squares approximation: y[t] =
$-0.855769 + 4.70249\,t + 5.55537\,t^2 - 0.0273601\,t^3$

```
Print["The velocity of the plane at t = 4.5 sec is y'[4.5] = ",
 y'[4.5] , " ft/sec."]
```

The velocity of the plane at t = 4.5 sec is y'[4.5] = 53.0387 ft/sec.