Orthogonal Matrices

Clear[u];

\[
\begin{bmatrix}
-6 & -3 & 6 & 1 \\
-1 & 2 & 1 & -6 \\
3 & 6 & 3 & -2 \\
6 & -3 & 6 & -1 \\
2 & -1 & 2 & 3 \\
-3 & 6 & 3 & 2 \\
-2 & -1 & 2 & -3 \\
1 & 2 & 1 & 6
\end{bmatrix}
\]

Note that the columns of \(u\) are orthogonal, and they are all of length 10.

\[
\text{Transpose}[u].u;
\%
\]

\[
\begin{bmatrix}
100 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 \\
0 & 0 & 100 & 0 \\
0 & 0 & 0 & 100
\end{bmatrix}
\]

Representation of Vectors in \(V = S \oplus S_{\text{perp}}\)

Let's find the projection of a vector \(y\) onto \(S = \text{span}(u_1,u_2,u_3)\).

Clear[y, yHat, u1, u2, u3];

\[
y = \{4, 3, 3, -1\};
\]

\[
u1 = \{1, 1, 0, 1\};
\]

\[
u2 = \{-1, 3, 1, -2\};
\]

\[
u3 = \{-1, 0, 1, 1\};
\]

Are the \(u\)'s mutually orthogonal?
Yes, the u's are mutually orthogonal.

Find the projection of y onto span(u1,u2,u3).

\[
y\text{Hat} = \frac{y.u1}{u1.u1} u1 + \frac{y.u2}{u2.u2} u2 + \frac{y.u3}{u3.u3} u3;
\]

Then y - y\text{Hat} is in \(U^{\perp}\)

... and the required decomposition is
Let's check that.
Is \( \hat{y} \) in \( \text{span}(u_1, u_2, u_3) \)?

\[
\text{RowReduce}[[u_1, u_2, u_3, \hat{y}]]; \\
\%
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Is \( \hat{y} - y \) in \( \text{span}(u_1, u_2, u_3)^\perp \)?

\[
\text{Transpose}[u].(y - \hat{y})
\]

\[
\{0, 0, 0\}
\]

Is \( y = \hat{y} + (y - \hat{y}) \)?
This one should be easy.

\[
y = \hat{y} + (y - \hat{y})
\]

\[
\text{True}
\]
Projection of a Vector onto the Column Space of an Orthogonal Matrix

Clear[u, y];

\[ u = \begin{pmatrix}
-6 & -3 & 6 & 1 \\
-1 & 2 & 1 & -6 \\
3 & 6 & 3 & -2 \\
6 & -3 & 6 & -1 \\
2 & -1 & 2 & 3 \\
-3 & 6 & 3 & 2 \\
-2 & -1 & 2 & -3 \\
1 & 2 & 1 & 6
\end{pmatrix}; \]

\[ y = \{1, 1, 1, 1, 1, 1, 1, 1\}; \]

Note that the columns of u are orthogonal, and they are all of length 10.

\[
\text{Transpose}[u].u; \\
\% // \text{MatrixForm}
\]

\[
\begin{pmatrix}
100 & 0 & 0 & 0 \\
0 & 100 & 0 & 0 \\
0 & 0 & 100 & 0 \\
0 & 0 & 0 & 100
\end{pmatrix}
\]

Therefore, the closest point to y in the column space of u is yHat, where

\[
(* \text{ yHat = } \\
1/100 \ ( (y.u1) u1 + \ldots + (y.u4) u4) = \ 1/10 \ u.\text{Transpose}[u].y *)
\]

Let's formalize that result. Let W be the column space of U.
See Lay 6.3, Theorem 10, p399.

\[
\text{projW}[y_] := 1/100 \ u.\text{Transpose}[u].y \\
yHat = \text{projW}[y]
\]

\[
\{6, 2, 6, 2, 6, 2, 6, 2, 2\}
\]

Check: Verify that yHat really is in the column space of u ...
Solve[u.{a1, a2, a3, a4} == yHat, {a1, a2, a3, a4}]

{[a1 -> 0, a2 -> 2/25, a3 -> 6/25, a4 -> 0]}

...and that y-yHat is orthogonal to each of the columns of u.

Transpose[u].(y - yHat)

{0, 0, 0, 0}

**QR Decomposition**

Clear[a, q, r];

a = {{3, 8}, {0, 5}, {-1, -6}};

Mathematica has built-in support for QR factorization, but its definition of the matrices involved differs somewhat from that of Lay. Consult *Mathematica*'s Help Browser for the function QRDecomposition, and see that entry's "Further Examples."

{q, r} = QRDecomposition[a];

Map[MatrixForm, {q, r}]

Note that the matrix Q of Lay is the matrix Conjugate[Transpose[q]] of Mathematica.
Q = Conjugate[Transpose[q]]; 
% // MatrixForm

\[
\begin{pmatrix}
\frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{35}} \\
0 & \frac{5}{\sqrt{7}} \\
-\frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{35}}
\end{pmatrix}
\]

Check.

\[a == Q.r\]
True

QR Decomposition

\[
a = \begin{pmatrix}
-10 & 13 & 7 & -11 \\
2 & 1 & -5 & 3 \\
-6 & 3 & 13 & -3 \\
16 & -16 & -2 & 5
\end{pmatrix};
\]

Mathematica has built-in support for QR factorization, but its definition of the matrices involved differs somewhat from that of Lay. Consult Mathematica's Help Browser for the function QRDecomposition, and see that entry's "Further Examples."

\[
{q, r} = \text{QRDecomposition}[a];
\]

\[
\text{Map[MatrixForm, \{q, r\}]}\
\]

Note that the matrix Q of Lay is the matrix Conjugate[Transpose[q]] of Mathematica.
Q = Conjugate[Transpose[q]];  
% // MatrixForm

\[
\begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{10} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{3}{10} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & 0 \\
\frac{4}{5} & 0 & \frac{1}{\sqrt{3}} & 0 \\
\frac{1}{10} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]

Check.

\[a == Q.r\]
True

**Least-Squares Solution of an Inconsistent System**

Find a least-squares solution to the inconsistent system \(ax=b\), where

\[
a = \begin{pmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix};
\]

\[b = \{2, 0, 11\};\]

Soln: Use the normal equations.

\[
\text{Transpose[a].a;}
\% // MatrixForm
\]

\[
\begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix}
\]
Now solve the system $a^T ax = a^T b$.

```math
\text{soln} = \text{Solve}[\text{Transpose}[a].x == \text{Transpose}[a].b, \{x1, x2\}][[1]]
```

Now solve the system $a^T ax = a^T b$.

```math
x = \{x1, x2\};
soln = Solve[Transpose[a].x == Transpose[a].b, \{x1, x2\}][[1]]
```

```math
x /. soln
```

\{x1 \rightarrow 1, x2 \rightarrow 2\}

### Least-Squares Solution of an Inconsistent System

Find a least-squares solution to the inconsistent system $ax=b$, where

```math
Clear[a, b, x, x1, x2, x3, x4];
```

```math
a = \begin{bmatrix} 
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
\end{bmatrix};
```

```math
b = \{-3, -1, 0, 2, 5, 1\};
```

Soln: Use the normal equations.
Now solve the system $a^T ax = a^T b$.

```
x = {x1, x2, x3, x4};
soln = Solve[Transpose[a].a.x == Transpose[a].b, {x1, x2, x3, x4}]
```

```
{(x1 → 3 - x4, x2 → -5 + x4, x3 → -2 + x4)}
```

Here is a clearer view of the solution.

```
{x1, x2, x3, x4} /. soln
```

```
{3 - x4, -5 + x4, -2 + x4}
```

and, better still, we have

```
(* x = {3, -5, -2, 0} + x4 {-1, 1, 1, 1} *)
```

**Least-Squares Error**

1. Find a **least-squares solution** to the inconsistent system $ax=b$, where
Clear[a, b, x, x1, x2, xHat, soln, d];

a = {{4, 0}, {0, 2}, {1, 1}};
b = {2, 0, 11};

soln: Use the normal equations.

Transpose[a].a;
% // MatrixForm

\[
\begin{pmatrix}
17 & 1 \\
1 & 5
\end{pmatrix}
\]

Transpose[a].b;
% // MatrixForm

\[
\begin{pmatrix}
19 \\
11
\end{pmatrix}
\]

Now solve the system \(a^T ax = a^T b\).

x = {x1, x2};
soln = Solve[Transpose[a].a.x == Transpose[a].b, {x1, x2}]

\[
\{\{x1 \to 1, x2 \to 2\}\}
\]

xHat is the least-squares solution. In this case it is unique because \(a^T a\) is invertible: \(xHat = (a^T a)^{-1}a^T b\).

xHat = x /. soln[[1]]

\[
\{1, 2\}
\]

(2) Find a least-squares error of this approximation.
Least-Squares Solution via QR Decomposition

Find a least-squares solution to the system \( ax=b \), where

\[
\begin{align*}
\text{Clear}[&a, b, q, r, Q, xHat, x1, x2, x3, soln]; \\
a &= \begin{pmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{pmatrix}; \\
b &= \{3, 5, 7, -3\};
\end{align*}
\]

Let's use the QR Decomposition of \( a \) to find \( xHat \), as in Lay Theorem 15, page 414.

(1) Calculate the QR Decomposition of \( a \).

\[
\{q, r\} = \text{QRDecomposition}[a]; \\
\text{Map}[\text{MatrixForm}, \{q, r\}]
\]

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{pmatrix},
\begin{pmatrix}
2 & 4 & 5 \\
0 & 2 & 3 \\
0 & 0 & 2
\end{pmatrix}
\]

Note that the matrix \( Q \) of Lay is the matrix \( \text{Conjugate}[\text{Transpose}[q]] \) of Mathematica.

\[
Q = \text{Conjugate}[\text{Transpose}[q]]; \\
\% // \text{MatrixForm}
\]

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}
\end{pmatrix}
\]

Check.
(2) Use the QR Decomposition of \(a\) to calculate \(\hat{x}\).

\[
x\hat{=} = \text{Inverse}[r].\text{Transpose}[Q].b
\]

\[
\{10, -6, 2\}
\]

Note that Lay suggests that, for numerical reasons, it is preferable to solve for \(\hat{x}\) directly from the equation \(r.\hat{x}=\text{Transpose}[Q].b\).

Let's try that.

\[
\text{soln} = \text{Solve}[r.\{x_1, x_2, x_3\} = \text{Transpose}[Q].b, \{x_1, x_2, x_3\}][[1]];
\]

\[
\{x_1, x_2, x_3\} = \{x_1, x_2, x_3\} /. \text{soln}
\]

\[
\{10, -6, 2\}\]