Lay, Section 4.6: Rank

These notes reflect material from our text, *Linear Algebra and Its Applications, Third Edition*, by David C. Lay, published by Addison Wesley, Boston, 2003.

Definitions

(1) row space, column space, and null space of the matrix A

(2) rank of the matrix A

(3) $S^{\perp}$, the space perpendicular to the set of vectors S

Results

Theorem. *If the matrices A and B are row equivalent, then they have the same row spaces. If the matrix B is in echelon form, then its nonzero rows form a basis for its row space, hence also for the row space of A.*

Theorem. *(Fundamental Theorem of Linear Algebra) The row space and the column space of an $m \times n$ matrix A have the same dimension. This common dimension is the rank of A, and it satisfies the equation*

$$rank\ A + dim\ null\ A = n.$$

Further characterization of invertible matrices (Lay, p. 267)

Theorem. *(Orthogonality Relations) Let $A \in \mathcal{M}_{m\times n}(\mathcal{R})$. Then*

$$null\ A = (row\ A)^{\perp} \text{ and } null\ A^{\perp} = (row\ A^{\perp})^{\perp} = (col\ A)^{\perp}.$$

Algorithms

Algorithm for computing a basis for the row space of a matrix A.

Caution: One has to be aware of the difference between rank and apparent rank of a matrix for matrices stored in a computer or calculator because real numbers are represented by “nearby” truncated numbers. The singular value decomposition is helpful in this regard for determining the rank of a matrix and also for calculating bases for the various spaces associated with the matrix, such as $null\ A, null\ A^{\perp}, col\ A$, and $row\ A$. See the Numerical Note in Lay, p268.

Exercises

We will solve some of the following exercises as a community project in class today. Finish these solutions as homework exercises, write them up carefully and clearly, and hand them in at the beginning of class next Friday. You are encouraged to use a computer algebra system whenever appropriate.

Exercises for Lay, Section 4.6, pp 269–271: 1, 3, 5, 7, 11, 15, 27, 28, 29, 30 (fundamental subspaces), 35