Algebra II: Review Exercises

These notes reflect material from our text, *A First Course in Abstract Algebra, Seventh Edition*, by John B. Fraleigh, published by Addison-Wesley, 2003.

Chapter 7. Advanced Group Theory

§*34. Isomorphism Theorems*:

Given G, H, K, calculate all sorts of things ($HK, H \cap K, ker\ \phi$, etc).

§*35. Series of Groups*:

What does it mean for a group to be *solvable*? What is the relevance of the Jordan-Hölder Theorem to the question of solvability of a group? Find a composition series for a specific group G (p319). Explain why the specific group G is or is not solvable.

§*36. Sylow Theorems*:

State three important properties of Sylow p-subgroups of a finite group G.

Exercise 36.6. Find two Sylow 2-subgroups of S_4.

Exercise 36.17. Show that G has a normal subgroup of order ....

§*37. Applications of the Sylow Theory*:

Exercise 37.17. Show that no group of order ... is simple.

§*38. Free Abelian Groups (and Classification of Finitely-generated Abelian Groups)*:

Exercise 38.10. Show that a free abelian group contains no elements of finite order.

Let $n = \ldots$. List all possible abelian groups of order n (up to isomorphism) and give the "prime-power" and torsion-coefficient decomposition of each such group.

§*39. Free Groups*:

Exercise 39.5. How many different homomorphisms are there of this specific group G onto or into this other specific group H?

§*40. Group Presentations*:

Construct a presentation for D_4, the group of rigid motions of the square, or for $\mathbb{Z}_4$, or for $\mathbb{Z}_3 \times \mathbb{Z}_3$. Indicate the logic behind your presentation.

Chapter 8. Groups in Topology

§*41. Simplicial Complexes and Homology Groups*:

Describe $C_i(X), Z_i(X), B_i(X)$, and $H_i(X)$ for the space X consisting of

the 0-simplex P_1,

the 1-simplex P_1P_2,

the 1-simplex $P_1P_2P_3$ (boundary of a triangle),

the 2-simplex $P_1P_2P_3$ (triangular surface),

the 2-simplex $P_1P_2P_3P_4$ (hollow tetrahedron),

the 3-simplex $P_1P_2P_3P_4$ (solid tetrahedron).

§*42. Computations of Homology Groups*:

Compute the homology groups of the space X consisting of

two tangent 1-spheres (figure eight),

happy face (circle with smile inside),

mickey mouse face (first draw mickey's face, then compute its homology),

all of the letters in the word "homology,"

coffee cup,

2-sphere wearing an inner tube around its equator,

the planet Saturn,

stack of three inner tubes,

stack of three donuts.

§*43. More Homology Computations and Applications*:

Illustrate the relationship between the Euler characteristic, $\chi(X)$, and the Betti numbers of $H_j(X)$, for one or more of the spaces X listed in the previous exercise.

Indicate how to calculate the homology of the real projective plane (Exercise 43.8), the homology of the Klein bottle (Example 43.1), and the homology of the Möbius strip (Example 43.5).

Exercises 42.12 and 43.9. Classify the closed 2-manifolds (both two-sided and one-sided), and indicate their homology groups.

Exercises 43.10–13. Given a map $f\colon X \to Y$ of the sort described in these exercises, compute the induced maps $f_{*n}\colon H_n(X) \to H_n(Y)$ for $n = 0, 1, 2$, by describing images of generators of $H_n(X)$.

State the Brouwer Fixed-Point Theorem, and sketch its proof. Relate figures 43.11 and 43.13 (pp. 376–377) to the proof of this theorem.

§*44. Homological Algebra*:

Exercise 44.2. Let A, B, and C be additive groups, and consider the exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0.$$

Draw the relevant conclusions (by considering exactness at A, B, and C, respectively). Prove some or all of your results.

Exercises 44.6–7 (and relevant details presented in the answer section). Compute the homology of the *pinched torus* by calculating the relative homology groups $H_n(X, a)$ for the torus X with subcomplex a (in the notation of figures 42.13–14, pp. 368–369).

Exercises 44.8–9 (and relevant details presented in the answer section). Compute the homology of the *pinched Klein bottle* by calculating the relative homology groups $H_n(X, a)$ for the Klein bottle X with subcomplex a (in the notation of figures 42.13–14, pp. 368–369).

Exercises 44.10–11 (and relevant details presented in the answer section). Compute the relative homology groups $H_n(X, Y)$ where X is the annulus of figure 42.11, and Y is the subcomplex consisting of the two boundary circles. What sort of *pinched annulus* have we constructed? What happens if we take Y to be just one of the boundary circles, or if we take Y to consist of the two boundary circles together with a disjoint, interior concentric circle?

Chapter 9. Factorization

§*45. Unique Factorization Domains*:

Exercise 45.10. Write $4x^2 - 4x + 8$ as a product of irreducibles in $\mathbb{Z}[x], \mathbb{Q}[x]$, and $\mathbb{Z}_{11}[x]$.

Exercise 45.12. Find all of the gcd's of 784, -1960, 448 in $\mathbb{Z}$. (See Exercise 46.10, below.)

Exercise 45.30. Prove that in a PID, every ideal is contained in a maximal ideal.

Exercise 45.34. Give an example of a ring which is noetherian but not artinian.

§*46. Euclidian Domains*:

Exercise 46.7–8. Calculate the positive gcd of 49,349 and 15,555 in $\mathbb{Z}$. Find $\lambda, \mu \in \mathbb{Z}$ so that this positive gcd can be written in the form $\lambda(49,349) + \mu(15,555)$.

Exercise 46.9. Calculate the gcd of two whopper polynomials in $\mathbb{Q}[x]$. (Use Mathematica.)

Exercise 46.10. Describe how to use the Euclidian Algorithm to find the gcd of n members of a Euclidian domain. Apply your method to Exercise 45.12, above. Does efficiency dictate that the integers be taken in any specific order?

Exercise 46.18. Prove that every field is a Euclidian domain.

§*47. Gaussian Integers and Multiplicative Norms*:

Exercises 47.1–4. Factor 5, 7, and $6 - 7i$ into irreducibles in $\mathbb{Z}[i]$.

Exercise 47.5. Find two distinct factorizations of 6 into a product of irreducibles in $\mathbb{Z}[\sqrt{-5}]$.

Exercise 47.7, 14. Use a Euclidian Algorithm in $\mathbb{Z}[i]$ to find a gcd of $8 + 6i$ and $5 - 15i$ in $\mathbb{Z}[i]$. How many gcd's are there of these two elements in $\mathbb{Z}[i]$?

Exercise 47.10 and Theorem 47.10. Which primes in $\mathbb{Z}$ remain irreducible in $\mathbb{Z}[i]$?

Chapter 10. Automorphisms and Galois Theory

§*48. Automorphisms of Fields*:

Exercises 48.1–8. Find all the conjugates of $\sqrt{2}$ over $\mathbb{Q}$, of $\sqrt{2} + i$ over $\mathbb{R}$, of $\sqrt{1 + \sqrt{2}}$ over $\mathbb{Q}(\sqrt{2})$.

Exercises 48.9–22. Describe all of the automorphisms of $G(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q})$. What is the structure of this group of automorphisms? Calculate the images of elements such as $\sqrt{2} + \sqrt{45}$ and $\sqrt{30}$ under some of these automorphisms. Calculate the fields fixed by certain subsets of this group of automorphisms.

Exercise 48.26. Describe the value of the Frobenius automorphism σ_2 on each element of the finite field with 4 elements. (See Example 29.19.) Find the fixed field of σ_2.

Exercise 48.36. Show that the p-th cyclotomic polynomial $\Phi_p(x)$ is irreducible. (See Corollary 23.17.) Carry out the program of Exercise 48.36. In particular, show that if $\alpha = a_1\zeta + a_2\zeta^2 + \cdots + a_{p-1}\zeta^{p-1}$ is left fixed by every automorphism in $G(\mathbb{Q}(\zeta)/\mathbb{Q})$, then $a_1 = a_2 = \cdots = a_{p-1}$, so $\alpha = -a_1$ lies in $\mathbb{Q}$.

Exercise 48.37. Suppose that $\alpha, \beta \in \mathbb{R}$ are both transcendental over F. Show that $F(\alpha)$ is isomorphic to $F(\beta)$.

Exercise 48.38. Let F be a field, and x an indeterminate over F. Determine all of the automorphisms in $G(F(x)/F)$.

§*49. The Isomorphism Extension Theorem*:

Exercises 49.1–3. Concerning extensions of certain automorphisms of subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

Exercises 49.4–6. Concerning zeros of $x^3 - 2$ in $\mathbb{C}$.

Exercise 49.7. Describe all the automorphisms of $\mathbb{Q}(\pi)$. *Hint:* This should remind you of fractional linear transformations. Now do Exercise 49.7.

Exercises 49.9–11. Concerning algebraic closures.

Exercise 49.13. Concerning the index of an extension, $\{E : F\}$.

§*50. Splitting Fields*:

Exercises 50.1–6. For each of the given polynomials, describe the splitting field E of the polynomial over $\mathbb{Q}$, and state its degree $[E\colon \mathbb{Q}]$.

Exercises 50.7–9. Calculate the order of each automorphism group.

Exercises 50.10. Calculate the splitting field of $f(x) = x^3 + x^2 + 1$ over $\mathbb{Z}_2$. How many elements does it contain. What can be said of the splitting field of *any* polynomial of degree three which is irreducible over $\mathbb{Z}_2$? How many splitting fields are there with exactly 8 elements? What polynomials do they split? (See section 33.)

Exercises 50.17. If E is a splitting field over F, and $[E\colon F]$ is finite, then E is the splitting field of a single polynomial.

Exercises 50.18. If $[E\colon F] = 2$, then E is a splitting field over F.

Exercises 50.20. $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ contains only the identity automorphism.

Exercises 50.21. Identify an automorphism group.

Exercises 50.22. Groups of automorphisms as permutation groups.

Exercises 50.23. Identify the splitting field of x^{p-1} over $\mathbb{Q}$. (See Corollary 23.17.)

§*51. Separable Extensions*:

Exercises 51.1–4. Find α such that the given field is $\mathbb{Q}(\alpha)$. Express the given generators in terms of α. Express α in terms of the given generators.

Exercise 51.9. Arithmetic combinations of separable elements are separable. State and prove a theorem on the "transitivity of separability."

Exercise 51.11. An algebraic extension of a perfect field is perfect.

Exercise 51.13. The separable closure of F in E.

Exercise 51.14. If E is a finite field of order p^n, then $G(E/\mathbb{Z}_p)$ is cyclic of order n with generator the Frobenius automorphism σ_p. (See Exercise 50.10 for specific examples.)

§*52. Totally Inseparable Extensions*:

§*53. Galois Theory*:

Exercises 53.1–8. The Galois correspondence.

Exercises 53.9–10, 12, 13. The structure of some specific Galois groups.

Exercises 53.11. The Galois correspondence for a specific splitting field.

Exercises 53.17–19. Norm and trace.

Exercises 53.23. Cyclic extensions.

§*54. Illustrations of Galois Theory*:

Exercise 54.2. Show how to calculate some of the primitive elements appearing in Figure 54.6. Calculate their irreducible polynomials.

Exercises 54.3–4. Calculate the splitting field K of $x^5 - 1$ over $\mathbb{Q}$. Describe the elements of $G(K/\mathbb{Q})$, and give the group and field diagrams for K over $\mathbb{Q}$. Describe the intermediate fields and corresponding intermediate groups. Calculate the irreducible polynomials for all elements appearing as generators of extension fields.

Exercise 54.9. Express the symmetric function $y_1^2 + y_2^2$ as a function of the elementary symmetric functions s_1, s_2.

Exercise 54.11. Prove that every finite group is isomorphic to some Galois group $G(K/F)$ for some finite normal extension K of some field F.

§*55. Cyclotomic Extensions*:

Let p be a prime. Show that the multiplicative group of the finite field $\mathbb{Z}_p$ is cyclic. Identify a generator of this cyclic group. Is the multiplicative group of the finite field with p^2 elements cyclic as well? If so, what are its generators? (For the structure of finite fields, see all of Section 33, Corollary 23.6, and Corollary 23.17. For the structure of infinite fields of characteristic p, see Section 55. For generators of finite cyclic groups, and subgroup diagrams for finite cyclic groups, see Corollary 6.16 and Example 6.17. What role do the primitive n-th roots of unity play in all of this? See Definition 55.2, Example 55.3, and Theorem 55.4.)

Exercises 55.1–8. Cyclotomic extensions, $\phi(n)$, $\Phi_n(x)$, constructible polygons, and splitting fields of $x^n - 1$ for $n = 6, 12$.

Exercises 55.10–12. Calculate Φ_n for $n = 1, \ldots, 6$ and for $n = 12$.

§*56. Insolvability of the Quintic*:

Exercise 56.8. Exhibit a specific polynomial of degree 5 in $\mathbb{Q}[x]$ which is not solvable by radicals over $\mathbb{Q}$.